Dynamics of Dissolved Vortices

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Introduction

Let $(L,H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be a Hermitian line bundle over a Riemann surface. A *vortex* is a Hermitian connection, $A \in \mathcal{A}^H(L)$, and a smooth section of L, $\phi \in \Omega^0(\Sigma; L)$, which minimise the *Yang-Mills-Higgs functional*,

$$E[A,\phi] := \frac{1}{2}\int_\Sigma |D_A\phi|_H^2 + |F_A|_H^2 + \frac{1}{2}\Big(\tau - |\phi|_H^2\Big)^2 \,\mathrm{dVol} \tag{1.1}$$

where $\tau > 0$ is a real parameter. Vortices are static minimisers of a similar time-dependent functional and so are sometimes called *static vortices*.

It is well known that $(A,\phi) \in \mathcal{A}^H(L) \times \Omega^0(\Sigma; L)$ is a vortex if and only if A is integrable (the $(0,2)$-part of its curvature vanishes) and if the pair satisfy the *vortex field equations* [5, pp. 54–55],

$$D_A^{0,1}\phi = 0 \tag{1.2}$$

$$F_A - \frac{\mathbf{i}}{2}(\tau - |\phi|_H^2)\omega^\Sigma = 0 \tag{1.3}$$

Moreover, Bradlow [2] showed that a necessary and sufficient condition for the existence of vortices is non-negativity of the quantity,

$$\varepsilon := \tau \operatorname{Vol}\Sigma - 4\pi N$$

where N is the degree of L.

In this poster we consider geodesic motion on the *moduli space* $\mathcal{M}^\varepsilon$ with respect to a natural Riemannian metric (in the manner of the geodesic approximation of Manton [6, §4.5]) and present some results the dynamics of dissolved ($\varepsilon = 0$) vortices when $N = 2$.

Vortex Moduli Spaces on Elliptic Curves

The moduli space of N-vortices is a quotient of a L^2-space by a proper group action, and so it is equipped with an L^2-metric, g^ε. However, $\mathcal{M}^\varepsilon$ is also in bijection with the space of degree N divisors on Σ, $\operatorname{Div}^N(\Sigma) \cong \Sigma^{(N)} := \Sigma^N/S_N$.

A linear equivalence class of divisors, $[D] := \{D' = D + (f)\}$, defines a finite dimensional subspace $H^0(\Sigma; \mathcal{O}(D)) \subset \Omega^0(\Sigma; L)$. The projectivisation of the latter is in bijection with $[D]$ and can be viewed as a subspace of $\mathcal{M}^\varepsilon$. $H^0(\Sigma; \mathcal{O}(D))$ also carries an L^2-metric from H, which then induces a metric, g^{FS}, on $\mathbb{P}H^0(\Sigma; \mathcal{O}(D))$.

A conjecture of Baptista and Manton [1] says that when Σ is the round sphere, g^ε is well approximated by g^{FS}. This was studied further by Manton and Romão [7] and Rink [8] and recently García Lara and Speight [4] proved a slight generalisation of the original conjecture.

In [3], the authors show that for an arbitrary Hermitian line bundle over a Riemann surface, the restriction of g^ε to a divisor class $[D]$ is well approximated by g^{FS} in C^1. If $[D]$ is fixed by an isomorphism, it defines a geodesic submanifold of $\mathcal{M}^\varepsilon$ and thus geodesics of g^ε are well approximated by geodesics of g^{FS} by the prior result.

Here we study divisors on $\Sigma = \mathbb{C}/\Lambda$ which are fixed by the isomorphism $z \mapsto -z$. We compute geodesics in terms of divisors and describe their topological properties, eventually using this description to compute their Higgs fields.

The Equivariant Weierstrass $\wp$ Function

For Σ an elliptic curve, the only divisors fixed by $z \mapsto -z$ are degree two divisors D linearly equivalent to $2 \cdot 0$. These are precisely the fibre of the two-to-one cover $\wp : \Sigma \to \mathbb{P}^1$. We consider the function $\wp_{\text{equiv}} = R \circ \wp$ for a Möbius transformation R such that $\wp_{\text{equiv}}$ is equivariant with respect to certain K_4-actions on Σ and $\mathbb{P}^1$. This allows us to pull-back great circles to geodesics on Σ.

We focus on two cases, the square and equianharmonic lattices, below we show stereographic plots of the critical values of $\wp_{\text{equiv}}$ in these cases.

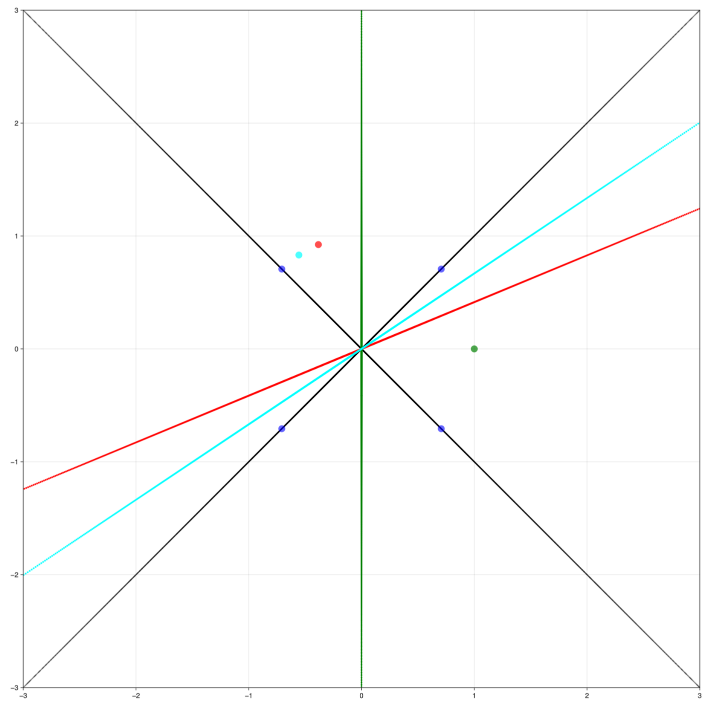

Figure 1: Critical values and loci for $\wp_{\text{equiv}}$ on the equianharmonic lattice with great circles plotted in colour.

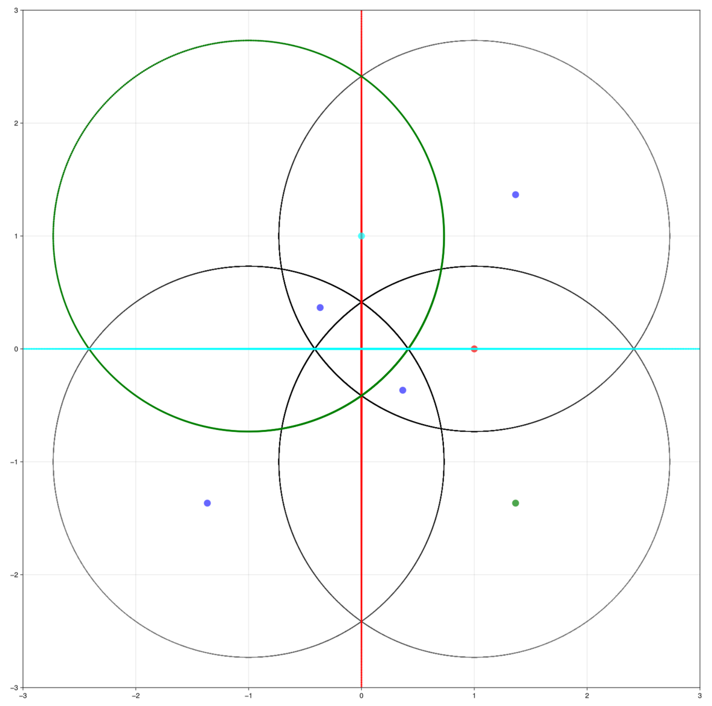

Figure 2: Critical values and loci for $\wp_{\text{equiv}}$ on the square lattice with great circles plotted in colour.

The critical values are plotted in blue and correspond to ramification values of $\wp_{\text{equiv}}$/collisions of vortex cores. Note that given $p \in \mathbb{P}^1$ we have a great circle $p^\perp := \{q \in \mathbb{P}^1 \mid q \cdot p = 0\}$. The black lines are *critical loci*. If a point lies on a critical locus, the great circle it defines passes through one of the critical values. The white regions therefore correspond to great circles which avoid critical values/collisions. Example great circles are shown in red, green, and cyan.

Geodesics

Plotting geodesics on the square lattice, we note that perturbing a geodesic within a region enclosed by critical loci does not change the homotopy type of the geodesic. Moving to a different region *does* change the homotopy type and so the regions correspond to different homotopy types of geodesics.

Moreover, we see that moving a great circle close to a critical value brings the geodesic close to a collision. This is further evidenced by animations of collisions.

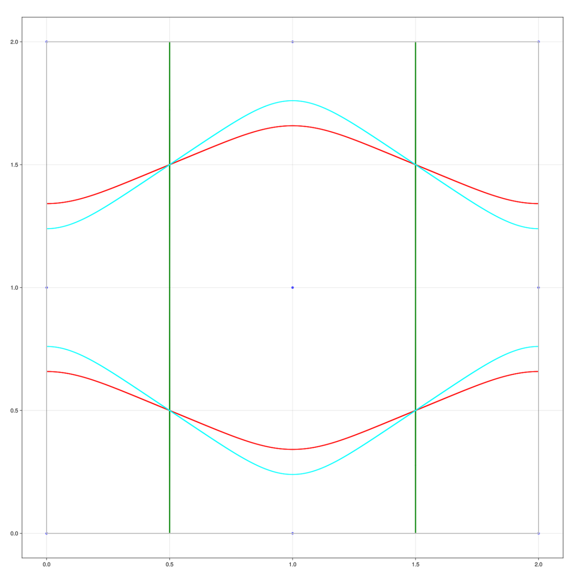

Figure 3: Geodesics on the square lattice.

The equianharmonic lattice differs from the square lattice in that it also supports contractible connected geodesics. Physically, these correspond to a pair of vortices circling a common barycenter.

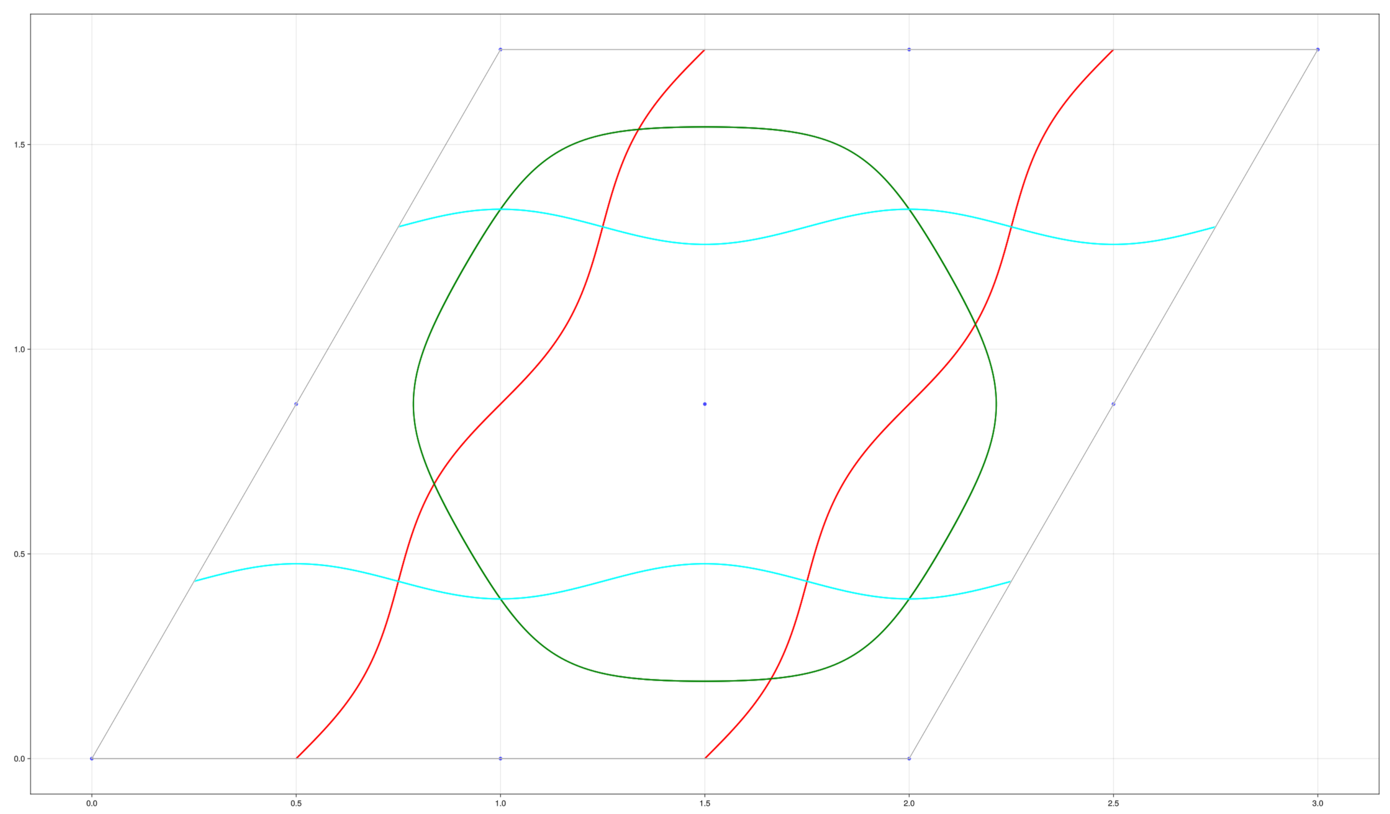

Figure 4: Geodesics on the equianharmonic lattice.

Conclusions and Future Work

We have demonstrated several possible dynamics of dissolved vortices and noted the existence of periodic motion of vortices. Moreover, we have begun to classify the topological types of possible dynamics and how this depends on the surface geometry, noting that certain types of vortex motion are only possible with certain base geometries.

Future directions include seeing whether this approach generalises to higher genus curves and attempting an algebraic classification of possible types of motion.

References

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Animations and high resolution plots are available at https://gchaudhuri.dev/gallery/gmom-2024/ or via the QR code below.