2-Vortices on Complex Toric Curves

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Outline

Vortices and their Moduli Spaces

Setup

- Fix a Hermitian line bundle over a (genus g) Riemann surface

$$(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma).$$

Let $N := \deg L = -\frac{\mathrm{i}}{2\pi}\int_\Sigma c_1(L)$. Assume $N > 0$.

- $(A, \phi) \in \mathcal{A}^H(L) \times \Omega^0(\Sigma; L)$: unitary connection and global section.
- $U \subset \Sigma$ a coord. patch,

$$D_A\phi\big|_U := \mathrm{d}\phi\big|_U + \mathrm{i}A_U\phi.$$

In particular $A_U \in \Omega^1(U; \mathbb{R})$.

The Yang-Mills-Higgs Functional

Consider

$$E[A,\phi] := \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{2}\|D_A\phi\|_{L^2}^2 + \frac{1}{2}\left\|\frac{1}{2}\left(|\phi|^2 - \tau\right)\right\|_{L^2}^2, \qquad (1)$$

where $\tau > 0$.

- *Vortices* are minimisers of this functional.
- Euler-Lagrange equations aren't sufficient to determine vortices.
- e.g. $F_{\hat{A}} \equiv C$, then $(\hat{A}, 0)$ solves the E-L equations, but (generally) not a minimiser.

Bogomolny Bound

Can rewrite [1], [2, p. 54]

$$E[A,\phi] = 2\left\|F_A^{0,2}\right\|_{L^2}^2 + \left\|D_A^{0,1}\phi\right\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda F_A + \frac{1}{2}\left(|\phi|^2 - \tau\right)\right\|_{L^2}^2 + \tau\int_\Sigma F_A. \tag{2}$$

- Last term is topological thus $E[A,\phi] \geq 2\pi\tau N$.
- So E minimised when other terms vanish, the energy of a vortex is precisely $2\pi\tau N$.

Field Equations

- $F_A^{0,2} = 0$ integrability: $\overline{\partial}_A := D_A^{0,1}$ defines a holomorphic structure for L.
- **The holomorphic structure of L is not fixed!**
- Other two yield the *vortex field equations*:

$$D_A^{0,1}\phi = 0, \tag{3}$$

$$\Lambda F_A + \frac{1}{2}(|\phi|^2 - \tau) = 0. \tag{4}$$

Gauge Action and Moduli Space

- Let $\mathcal{V}$ be the set of solutions to the field equations.
- Field equations invariant under gauge action

$$h \cdot (A, \phi) := (A - \mathrm{d}\chi\,, e^{\mathrm{i}\chi}\phi),$$

  where $\chi \in \mathcal{G} := \Omega^0(\Sigma; \mathbb{R})$.
- Let $\mathcal{M} := \mathcal{V}/\mathcal{G}$ be the *vortex moduli space*.
- When is $\mathcal{V}$ (and thus $\mathcal{M}$) non-empty?

Structure of the Moduli Space

The Dissolving Limit

- Integrating $\Lambda F_A + \frac{1}{2}\left(|\phi|^2 - \tau\right) = 0$ yields a necessary condition for $\mathcal{M} \neq \varnothing$,

$$\varepsilon := \tau \operatorname{Vol} \Sigma - 4\pi N \geq 0. \tag{5}$$

  Bradlow [1] noted that this is in fact sufficient.
- Noting that $\varepsilon = \|\phi\|_{L^2}^2$, $\varepsilon = 0$ implies $\phi = 0$.
- Only d.o.f. is in gauge field which must satisfy $F_{\hat{A}} \equiv C$.
- When $\varepsilon = 0$, say vortices are *dissolved*, limit $\varepsilon \to 0$ is the *dissolving limit*.

Sketch proof of Bradlow's argument i

Idea of [1] is to view (A, ϕ) as being determined by algebraic data.

- Choose a complex structure on L (a $\mathbb{C}^\times$-g.e.c. $[\overline{\partial}_A]$).
- Choosing $\overline{\partial}_A$ uniquely determines D_A (Chern connection), pick $\overline{\partial}_{\hat{A}}$ s.t. $F_{\hat{A}} \equiv \frac{2\pi N}{\mathrm{Vol}\,\Sigma}$ (unique up to U(1)-gauge).
- Choose $[\hat{\phi}] \in (\ker \overline{\partial}_{\hat{A}})/\mathcal{G}$ s.t. $\left\|\hat{\phi}\right\|_{L^2}^2 = 1$.

Sketch proof of Bradlow's argument ii

- Let u solve

$$\Delta u - \frac{\varepsilon}{\mathrm{Vol}\,\Sigma} + \varepsilon \left|\hat{\phi}\right|^2 e^u = 0. \tag{6}$$

  N.B. This has solutions iff (5) is satisfied.

- Then

$$(A, \phi) = \left(\hat{A} + \frac{1}{2} * \mathrm{d}u\,, \sqrt{\varepsilon} e^{u/2} \hat{\phi}\right)$$

  solves the vortex field equations.

- Construction is gauge invariant, yields a well defined g.e.c. $[A, \phi] \in \mathcal{M}$.

Structure of the Moduli Space

- When $\varepsilon > 0$, elements of $\mathcal{M}$ and pairs $([\overline{\partial}_{\hat{A}}], [\hat{\phi}])$ are in bijection.
- Pair $([\overline{\partial}_{\hat{A}}], [\hat{\phi}])$ is the data of a degree N *(effective) divisor* on Σ, i.e.

$$D = \sum_{p \in \Sigma} a_p [p],$$

  s.t. $a_p \in \mathbb{Z}_{\geq 0}$, $\sum a_p = N$.
- Thus $\mathcal{M} \leftrightarrow \mathrm{Div}^N(\Sigma) \leftrightarrow \Sigma^N / S_N$.
- $\mathcal{M}$ can be given a smooth structure [3], [4].

The Abel-Jacobi map

- When $\varepsilon = 0$, $[A, \phi] = [\hat{A}, 0]$, thus $\mathcal{M} \cong \mathrm{Pic}^0(\Sigma)$.
- When $\varepsilon > 0$, we have a forgetful map $\mathrm{AJ} : \mathcal{M} \to \mathrm{Pic}^0(\Sigma), [A, \phi] \mapsto [\overline{\partial}_A]$.
- On divisors, this is the usual Abel-Jacobi map. Thus $\mathrm{AJ}^{-1}([\overline{\partial}_A]) \cong \mathbb{P}^k$.
- If $N + 2 - 2g \geq 0$, $k \equiv N - g$ and $\mathcal{M}$ is a $\mathbb{P}^{N-g}$-bundle over $\mathrm{Pic}^0(\Sigma)$.

The Vortex Metric

Definition

- For any $\gamma : (-\epsilon, \epsilon) \to \mathcal{V}$, $\|\dot{\gamma}_v(0)\|_{L^2}^2$ is well defined.
- Given $\gamma_v : (-\epsilon, \epsilon) \to \mathcal{M}$ representing $v \in T_p\mathcal{M}$, choose lift $\tilde{\gamma} : (-\epsilon, \epsilon) \to \mathcal{V}$ orthogonal to the action of $\mathcal{G}$.
- Define

$$\|v\|^2 := \left\| \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \tilde{\gamma} \right\|_{L^2}^2$$

- Define

$$g^{\mathcal{M}}(u, v) := \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2)$$

Properties

- $(\mathcal{M}, g^{\mathcal{M}})$ is Kähler, volume is known Baptista [5].
- No global closed form expression for $g^{\mathcal{M}}$ (local formula due to Samols [6]).
- Is it possible to approximate the metric in the dissolving limit?

The Dissolved Metric and Fibres of AJ

- For some complex structure $[\overline{\partial}_A]$, $V := (\ker \overline{\partial}_A)/\mathcal{G} \cong \mathbb{C}^{N-g}$.
- Given $[\hat{\phi}] \in V$, $\left\|\hat{\phi}\right\|_{L^2}^2$ is well defined.
- Thus $\mathbb{P}V \cong \mathsf{AJ}^{-1}([\overline{\partial}_A])$ is equipped with a natural Fubini-Study metric.

C^1 convergence in the Dissolving Limit

Theorem (C., Harland, and Speight)

Let $\tilde{g}$ be the restriction of $g^{\mathcal{M}}$ to some fibre of AJ *and let g^{FS} be the natural Fubini-Study metric on the fibre. Then there is a constant C, depending on $(L, H) \to (\Sigma, g^{\Sigma})$ such that*

$$\left\| \frac{1}{\varepsilon}\tilde{g} - g^{\mathrm{FS}} \right\|_{C^1} < C\varepsilon. \tag{7}$$

Consequences

- C^0-convergence is enough to guarantee convergence of spectra of Laplacians [7].
- If $\mathrm{AJ}^{-1}([\overline{\partial}_A])$ is geodesically closed, C^1-convergence ensures C^0-convergence of RHS of the geodesic equation.
- Enough to guarantee uniform convergence of geodesics [8, p. 58].

2-Vortices on Complex Toric Curves

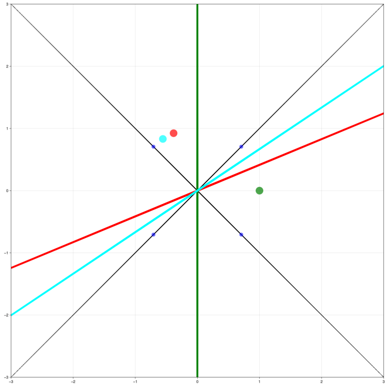
Complex Toric Curves

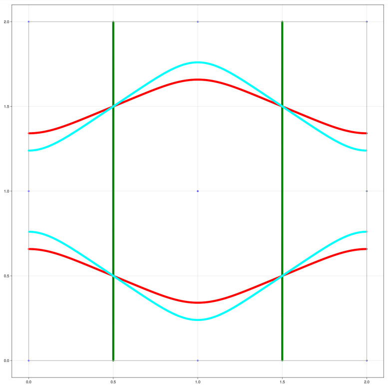
- Given some lattice $\Lambda = \mathbb{Z}\{2\omega_1, 2\omega_3\} \subset \mathbb{C}$, $\Sigma = \mathbb{C}/\Lambda$ is a *complex toric curve*.
- If $N = 2$, $\mathcal{M}$ is a $\mathbb{P}^1$-bundle over $\mathrm{Pic}^0(\Sigma) \cong \Sigma$.
- $\mathrm{AJ} : \Sigma^2/S_2 \to \Sigma, \{p, q\} \mapsto p + q$.
- Clear that $\mathrm{AJ}^{-1}(0)$ is fixed by the isometry $\{p, q\} \mapsto \{-p, -q\}$, thus it must be geodesically closed.
- Thus we can approximate geodesics in $\mathrm{AJ}^{-1}(0)$ by geodesics on $\mathbb{P}^1$.

Identifying Fibres with $\mathbb{P}^1$

- $\mathrm{AJ}^{-1}(0) = \{\{p, -p\} \mid p \in \Sigma\} \cong \Sigma/\mathbb{Z}_2$
- Equip Σ with the K_4 action generated by $z \mapsto z + \omega_1$ and $z \mapsto z \mapsto \omega_3$.
- This is an action by isometries on Σ exchanging $\omega_0 = 0, \omega_1, \omega_3$, and $\omega_2 = \omega_1 + \omega_3$.
- There is a unique (up to conjugation) action by isometries on $(\mathbb{P}^1, g^{\mathrm{FS}})$ (representation theory).
- Need to find $f : \Sigma \to \mathbb{P}^1$ which is even and equivariant with respect to the K_4 action.

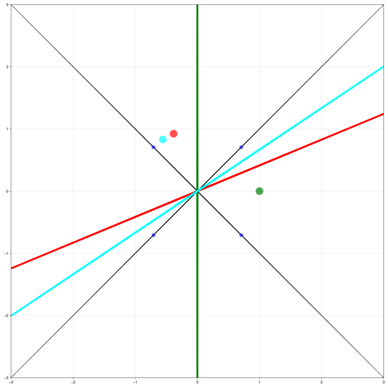
Recovering Geodesics

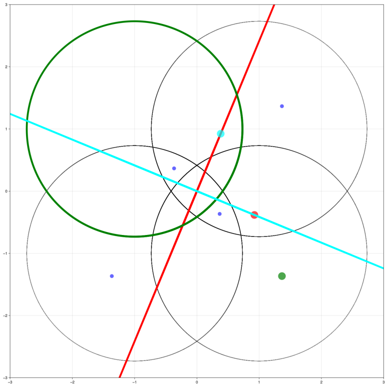




$$\Sigma = \mathbb{C}/\mathbb{Z}\{2 \cdot 1, 2 \cdot \mathrm{i}\}$$

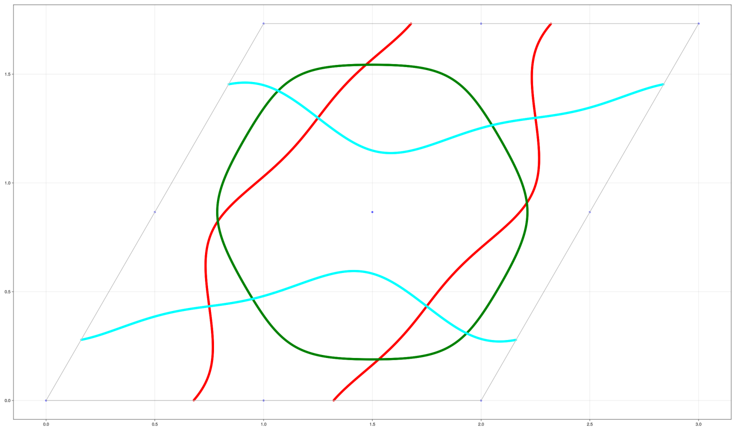
Lattice Dependence and Scattering i





$$\Sigma = \mathbb{C}/\mathbb{Z}\{2 \cdot 1, 2 \cdot \mathrm{i}\} \qquad \Sigma = \mathbb{C}/\mathbb{Z}\left\{2 \cdot 1, 2 \cdot e^{\mathrm{i}\pi/3}\right\}$$

Lattice Dependence and Scattering ii



$$\Sigma = \mathbb{C}/\mathbb{Z}\left\{2\cdot 1, 2\cdot e^{\mathrm{i}\pi/3}\right\}$$

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