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Dynamics of Dissolved Vortices

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Recap/Notation

Abelian Higgs vortices (will always be static, BR crit. coupled & BPS for this talk) on a cmpct. Riem. surf. Σ form a moduli space

$$M_N \cong \Sigma^{(N)} := \Sigma^N / S_N$$

↑ locations of the zeros of the Higgs field

"First order dynamics" M_N carries a natural metric, g_M, & geodesics of this carry info about slowly moving vortices.

Problem: No global explicit form for this metric.

Solns

- Samols – Strachan localisation
  - pro: good for numerics, convenient for proofs about general facts
  - con: not global & only semi-explicit
- Parametric Approximation
  - pro: can be global, sometimes provides analytic approximations
  - con: need to prove analytic (in)validity, often bespoke

e.g. (of latter) when $|\Sigma| \longrightarrow \infty$, $(M_N, g_M) \longrightarrow (\Sigma^N, (g_\Sigma)^N)$

[Nagy 2017 - "Berry conn. of G-L vortices"]

~~Point of~~ Theme of talk: what happens when $|\Sigma| \longrightarrow |\Sigma|_{crit}^{+}$

$|\Sigma| \longrightarrow |\Sigma|_{crit}$?

Notation

Σ is some cmplt. Riem. surf [For majority of talk can take $\Sigma \cong S^2$] Metric might not be round!!

~~A~~ ~~$D_A \in \mathcal{A}(L,H)$ is a~~ $(L,H) \longrightarrow \Sigma$ is a line bundle.

$\phi \in \Omega^0(\Sigma; L)$ is a Higgs field

$D_A \in \mathcal{A}(L,H)$ is a unitary conn. $\quad \langle \phi, \psi \rangle = \mathrm{Re}\, \phi^* \psi$

$(B=)F_A = D_A \circ D_A$ is the curvature 2-form
$\in \Omega^2(\Sigma; \mathbb{R})$

[$\phi : \Sigma \to \mathbb{C}$, $\phi = \{\phi_N : \Sigma \setminus \{South\} \to \mathbb{C}$, $\phi_S : \Sigma \setminus \{North\} \to \mathbb{C}\}$

$D_A|_U = d + iA_U$, $\quad A_U \in \Omega^1(\Sigma, \mathbb{R})$]

(static, crit-coupled, BPS) vortex is a sol^n to

$$D_A^{0,1}\phi = 0 \qquad (V1)$$

$$F_A = \tfrac{1}{2}(\tau - |\phi|^2)\,\omega_\Sigma \qquad (V2)$$

Key obs. [Bradlow '89, García-Prada '92]

Vortices exist if and only if:

$$\varepsilon := \tau \operatorname{Vol}\Sigma - 4\pi N \geq 0$$

Arrived at by integrating (V2) $\rightsquigarrow \varepsilon = \|\phi\|_{L^2}^2$.

At $\varepsilon = 0$, $\phi \equiv 0$ so vortex is "dissolved", $\varepsilon \to^+ 0$ is the dissolving limit.

Above this limit, $M_N \cong \Sigma^{(N)} \cong \mathbb{CP}^N$. not the obvious map. will say more on this later.

M_N carries a "natural" metric arrived at by restricting kinetic energy term to orbits orthog. to gauge action.

This is g_{M} the *vortex metric. define using curve $(A(t), \phi(t)) \in M_N$ w/ gauge fixing

$d^*\dot{A} = \langle \dot{\phi}, i\phi \rangle$ Will use this for g_{FS} later

$\operatorname{div}\dot{A} = \langle \dot{\phi}, i\phi \rangle$

~~Pseudovortices~~ / Holomorphic Pairs

In general, hard to find explicit (A,ϕ) (or indeed g_m)

[Baptista & Manton ~~2~~2003] suggest looking at simpler eqn^s.

$$D^{0,1}_{\hat{A}}\hat{\phi} = 0 \qquad (P1)$$

$$F_{\hat{A}} = \frac{4\pi N}{\mathrm{Vol}\,\Sigma}\,\omega_\Sigma \qquad (P2)$$

$$\|\hat{\phi}\|^2_{L^2} = 1 \qquad (P3)$$

Note: ~~$(\hat{A},\hat{\phi})$~~ $(\hat{A}, \sqrt{\varepsilon}\,\hat{\phi})$ then solves (V1) & (V2) "on average".

These are ~~easy(er) to~~ so easier to solve.

- For $\Sigma \cong S^2$, there is only one $\hat{A}$ solving (P2) up to ~~reg.~~ local gauge.
- Fix such an $\hat{A}$, ~~there~~ soln^s to (P1) form an N-dimensional cmplx vector space.
- (P3) reduces this to $S^{2N+1} \subseteq \mathbb{C}^N$.
- global phase reduces this to $\mathbb{CP}^N$.

~~[Historical note: This is precisely~~

(*) [Historical note: This is precisely what Bradlow does to find vortex soln^s on a general Σ.

Soln^s to (P1–3) are called pseudovortices by B&M but Bradlow (and others after him) refer to them as holomorphic pairs.

This is the diffeomorphism $M_N \longrightarrow \mathbb{CP}^N$

$$\begin{array}{ccccc} M_N & \longrightarrow & \Sigma^{(N)} & \longrightarrow & \mathbb{CP}^N \\ \phi & \longmapsto & \phi^{-1}(0) & \longmapsto & \hat{\phi} \text{ s.t. } \hat{\phi}^{-1}(0) = \phi^{-1}(0) \end{array}$$

Bradlow directly relates ϕ & $\hat{\phi}$ by way of an elliptic PDE.]

~~[García Lara & Speight 2022]~~ ok

Mod. sp. of soln^s. to (P1−3) has a nat. L^2-metric, $\Omega^0(\Sigma; L)$. Let $\hat{M}_N$ be the mod. sp. of soln^s. to (P1−3),
$\hat{M}_N \cong \mathbb{CP}^N$

g_M $g_{\hat{M}} \cong g_{FS}$ (gauge quot. of $\|\cdot\|^2_{L^2}$ on $\Omega^0(\Sigma; L)$.) ~~use current~~ Define using curve $(\hat{A}(t), \hat{\phi}(t)) \in \hat{Z}_N$ w/ gauge orthog

Fubini-Study metric.

Conj: [Baptista & Manton 2003]

$\boxed{\text{As } \varepsilon \to 0, \tfrac{1}{\varepsilon} g_{M_N} \to g_{FS}}$ [~~ok~~ initially for $\Sigma = S^2$, nat. exts. to higher genus.]

Progress:

[Rink 2014]: Asymptotic arg. for arb. genus, studied in hyperelliptic case.

[García Lara & Speight 2022]: Analytic proof and proof that for $\Sigma \cong S^2$ (not nec. round), $\|\tfrac{1}{\varepsilon} g_M - g_{FS}\|_{C^0} < C\varepsilon$. ($C$ depends on (Σ, g_Σ) & N)

Import ~~core~~: [GL&S]

~~$\|\nabla\varepsilon\|$ Lemma [GL&S]~~

~~$\| |\phi| / \sqrt{\varepsilon} - |\hat{\phi}| \|$~~

Prop: [GL&S]

$(\hat{A}, \hat{\phi})$ solves (P1−3), let $u \in C^\infty(\Sigma)$ be the ! sol^n to

$$\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon e^u |\hat{\phi}|^2 = 0$$

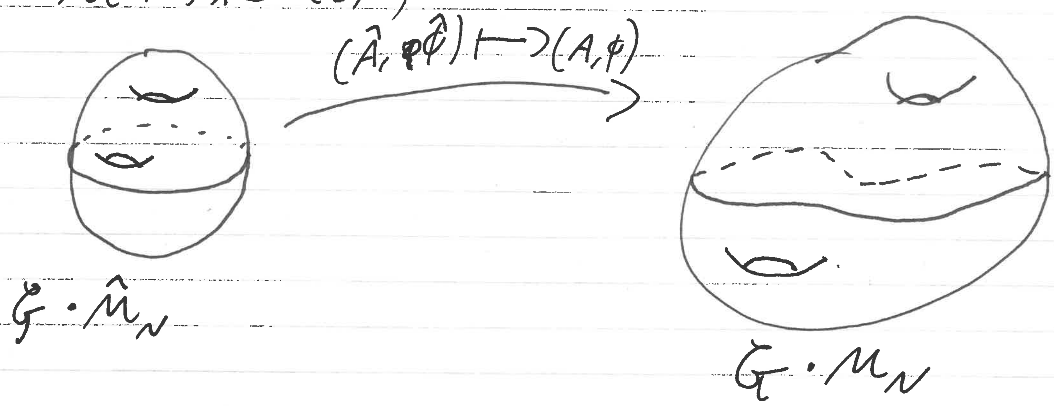
then $(\hat{A} - \tfrac{1}{2} * du, \sqrt{\varepsilon}\, e^{u/2} \hat{\phi})$ solves (V1−2).

All such soln^s. of (V1−2) arise in this way.

This yields a diff. $\hat{M}_N \xrightarrow{\sim} M_N$.

Picture

$\mathcal{C} = \mathcal{A}(L,H) \times \Omega^0(\Sigma; L)$



Lemma [GL&S] $\exists\, C > 0$ dep. on (Σ, g_Σ), N

$\forall\, (\hat A, \sqrt{\varepsilon}\,\hat\phi) \in \hat{\mathcal{V}}_N$

$$\left\| \frac{|\phi|}{\sqrt{\varepsilon}} - |\hat\phi| \right\|_{C^0} < C\varepsilon$$

$$\| F_A - F_{\hat A} \|_{C^0} < C\varepsilon$$

*upshot: "for a fixed charge/genus, pseudovortices $\approx$ approx. vortices"

Thm [GL&S] $\exists\, C > 0$ dep. on (Σ, g_Σ), N,

$$\left\| \tfrac{1}{\varepsilon} g_M - g_{FS} \right\|_{C^0} < C\varepsilon$$

[will talk about proof in part section on C^1-conv.]

pw difference in resp. cmpts. bdd. by $C\varepsilon$

C^1-convergence

Thm [C., Haaland & Speight]

For any (Σ, g_Σ), for any $N > 0$, there exists $C > 0$ s.t.

$$\left\| \tfrac{1}{\varepsilon} g_M - g_{FS} \right\|_{C^1} < C\varepsilon.$$

Proof

- have ~~metrics~~ spaces (M_N, g_M), $(\hat M_N, g_{\hat M})$
- Fix gauge $\hat A$ $\rightsquigarrow$ $\hat M_N \cong \mathbb{CP}^{2N}$ & $\hat\phi$ s.t. $\bar\partial^{0,1}_{\hat A}\hat\phi = 0$ is f.d. vec. sp.
- Suppose $\{s_\alpha\}$ is some L^2-orth. basis, $\hat M_N = S^{2N+1}/U(1)$, $S^{2N+1} \subset \mathbb{C}^{N+1}$
- ~~$\hat M_N = \{\Sigma_\alpha x^\alpha s_\alpha \mid \Sigma_\alpha |x^\alpha|^2 = 1\}/U(1)$~~

- $g_{\hat{\mathcal{M}}}$ is just the obv. quotient metric on this space (orth cond. $\langle \dot{p}, i p\rangle = 0$)

- pull-back $g_{\mathcal{M}}$ to $\hat{\mathcal{M}}_N$ by diff ~~(A) (complex~~

~~$\frac{1}{\varepsilon} g_{\mathcal{M}} = g_{\hat{\mathcal{M}}} + O(\alpha)$~~

- $\frac{1}{\varepsilon} g_{\mathcal{M}}\Big|_{(\hat{A},\hat{\phi})} = g_{\hat{\mathcal{M}}}\Big|_{(\hat{A},\hat{\phi})} + O\big(u(\hat{\phi}), \dot{u}(\hat{\phi}), \dot{n}(\hat{\phi})\big)$

use analytic tools to bound these in particular a lemma from GL$S

Subtleties & Consequences

- For C^0, it's enough to bd. principle cmpnts. For C^1 we need metric cmpnts. ~~C^2 likely to be n~~

- Consider the geodesic eq^n.

$$\ddot{\gamma}^\lambda = -\Gamma^\lambda{}_{\mu\nu}\, \dot{\gamma}^\mu \dot{\gamma}^\nu$$

$\Gamma^\lambda{}_{\mu\nu}$ are (essentially) $\partial^\lambda g_{\mu\nu}$, so coeffs. of Γ for $\frac{1}{\varepsilon} g_{\mathcal{M}}$ are C^1 close to coeffs for $g_{\hat{\mathcal{M}}}$.

~~Γ for~~

$\implies$ ~~Sols. to the IVP for geodesics are close to~~

ODE theory

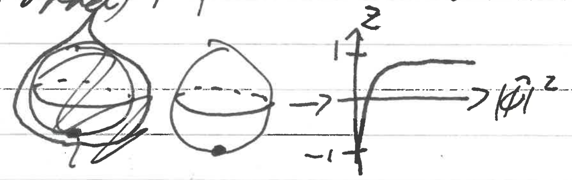
geodesics of $g_{\mathcal{M}}$ are well approximated by geodesics of $g_{\hat{\mathcal{M}}}$.

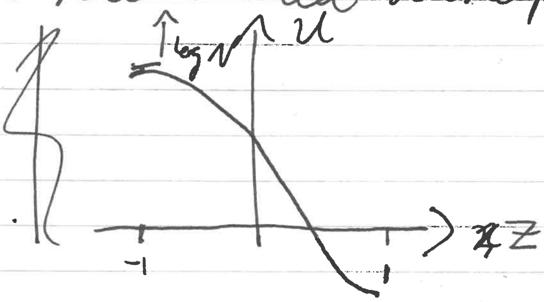
- This does generalise to ~~higher genus of~~ Σ with higher genus (almost immediately). *Modulo some minor technicalities (will mention later) the proof is almost the same.

Rate of growth of C as a function of N

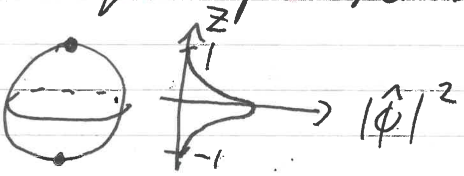
- ~~Namely~~ C depends on $\|u\|_{C^0}$ $\left[\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon e^{u}|\hat{\phi}|^2 = 0\right]$
- Depending on the choice of $\hat{\phi}$, u can behave badly.

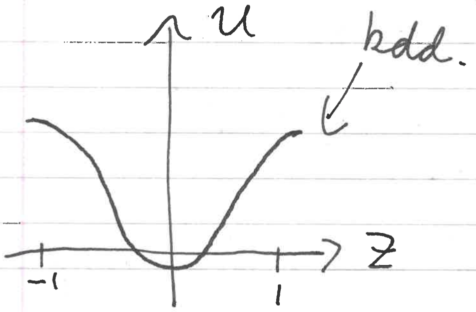
e.g. $\hat{\phi}$ s.t. $\Sigma = S^2$, $(\Sigma, g_\Sigma) = (S^2, g_{round})$, $\hat{\phi}$ s.t. zeros are concentrated at one pole

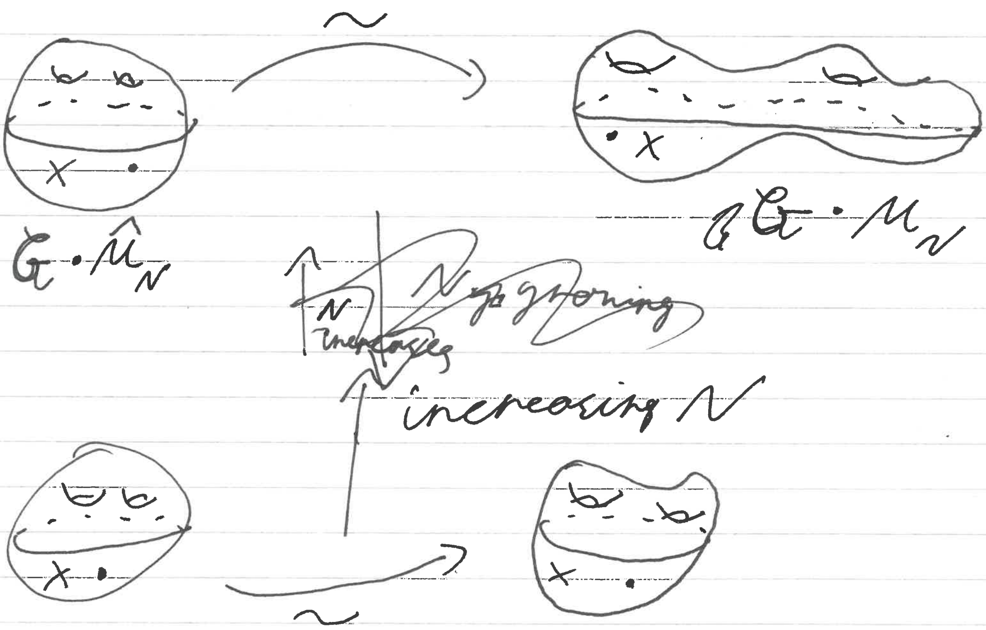




is zeros of $\hat{\phi}$ are equally distributed at either pole







2-Vortices on Flat Tori

When Σ has non-zero genus, $\mathcal{M}_N \not\cong \mathbb{C}P^N$

$$\hat{\mathcal{U}}_N // S_N \quad \mathcal{M}_N \cong \Sigma^{(N)}$$

~~Instead~~ This is due to the fact that there is a choice of ~~A~~ D_A s.t. $F_{\hat{A}} = \frac{2\pi N}{|\Sigma|}$ (even modulo gauge).

~~C' result is still true, but const. C is only valid for each choice.~~

~~For each choice of $D_{\hat{A}}$, we have $\operatorname{Ker} D_{\hat{A}}^{0,1} \cong \mathbb{C}^{R+1}$~~

For each choice of $D_{\hat{A}}$, we have

$$\operatorname{Ker} D_{\hat{A}}^{0,1} \cong \mathbb{C}^{R+1}$$

~~$R \neq N$~~

R is not necc. $= N$!!! it might even change depending on $D_{\hat{A}}$!

but for ~~case~~ $\Sigma = \mathbb{C}/\Lambda$, $R \equiv N-1$.

Fact: $\{D_{\hat{A}} \mid F_{\hat{A}} = \frac{2\pi N}{|\Sigma|}\} / \text{gauge} \cong \Sigma$

"~~Proof~~" construction: $\{p_1, \dots, p_N\} \in \Sigma^{(N)}$

$$P_{\hat{A}} = p_1 + p_2 + \cdots + p_N$$

$\uparrow$ this uniquely determines $D_{\hat{A}}$

~~Consider~~

consider: $\{\{p_1, p_2\} \in \Sigma^{(2)} \mid p_1 + p_2 = 0\}$

2-Vortices on Flat Tori

When genus of $\Sigma \neq 0$, $M_N \not\cong \mathbb{CP}^N$

$$\Sigma^{(N)} \cong M_N \cong \hat{M}_N$$

This is b/c there isn't a unique (/gauge) $D_{\hat{A}}$ s.t. $F_{\hat{A}} = \frac{2\pi N}{|\Sigma|}$.
There is a choice.

For each choice of $D_{\hat{A}}$, we have

$$\ker D^{0,1}_{\hat{A}} \cong \mathbb{C}^{k+1}$$

but k is not nec. $= N$! It might change depending on $D_{\hat{A}}$!

So ~~two~~ problems:

1 – Does choice of $D_{\hat{A}}$ affect proof of C^1-cvgnce?
2 – Do we still have convergence of geodesics?

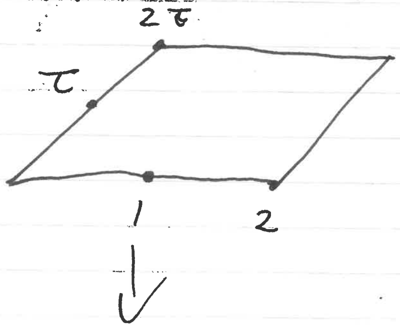
Answer to 1: yes, but not ~~terribly~~ hugely.
bound $\|\frac{1}{\varepsilon}g_\varepsilon - g_{FS}\|_{C^1} < C\varepsilon$ not true across all of $\hat{M}_N$,
but only on ~~on~~ subspaces, $(D_{\hat{A}}, \&\sqrt{\varepsilon}\phi)$ for fixed
but arbitrary $D_{\hat{A}}$. (call this a "fibre of the Abel-Jacobi map")

Answer to 2: In general no. If this subspace
is totally geodesic (geodesics stay on the subspace)
then yes, however.

One such scenario is ~~the subsp~~ relative motion
of 2-vortices on a flat torus.

Consider $\{P_1, P_2\} \subset \Sigma$, $\Sigma = \mathbb{C}/\Lambda$

Let $\{p, q\} \subset \Sigma$ be s.t.
$p + q = 0$

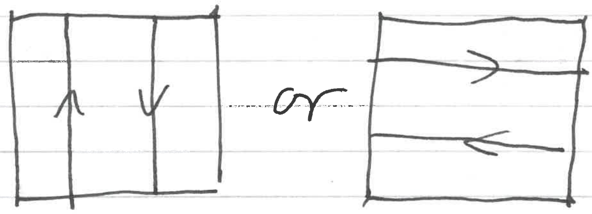


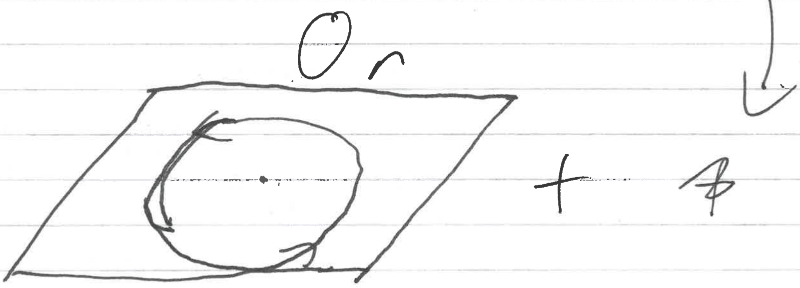
can show topologically
that space of such
configs is $\cong \mathbb{CP}^1$

It can be shown that this $\mathbb{CP}^1$ is ~~one~~ a fibre of the ~~ab~~ AJ map & it is fixed by rotations of π so it is ~~g~~ totally geodesic.

$\Rightarrow$ We can approx. vortex metric geods. by Fubini-Study metric geods.

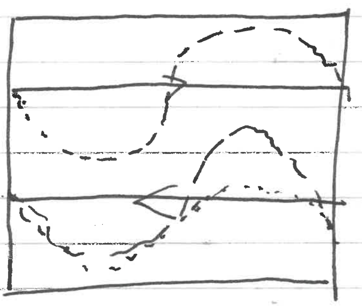
Boils down to inverting some elliptic function $f: \Sigma \longrightarrow \mathbb{CP}^1$, but we have trajectories





should be a rounded hexagon.

& equianharmonic case is generic.



Ex: Exact approximation to 2-vortex dynamics.