Vortices near the Bradlow Limit

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What are Vortices?

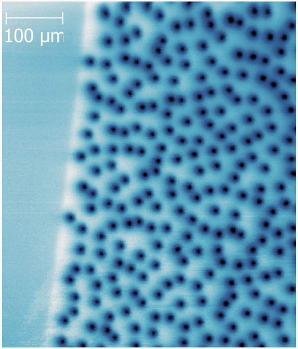

Figure 1: Vortices in a superconductor film (Wells et al., 2015)

- Circular current flows in superconductors
- Described mathematically by *Ginzburg-Landau theory*
- Geometric generalisations of G-L theory have ties to:
  - Vector bundle stability
  - Hamiltonian Gromov-Witten invariants

The Functional

- Two fields,
  - *Higgs field*: $\phi \in C^\infty(\mathbb{R}^2, \mathbb{C})$
  - *Gauge field*: $A_i \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$
- Differential terms,
  - *Covariant derivative*: $D_i\phi = \partial_i\phi - \mathrm{i}A_i \cdot \phi$
  - *Curvature*: $B_{ij} = \partial_i A_j - \partial_j A_i$
  - Write $B^2 = (B_{12})^2$
- The *Ginzburg-Landau functional*,

$$E(A_i, \phi) = \frac{1}{2}\int B^2 + \overline{D_i\phi}D_i\phi + \frac{\lambda}{4}(\tau - \overline{\phi}\phi)^2 \, \mathrm{d}^2x$$

where $\lambda, \tau > 0$ (interested in $\lambda = 1$)

Vortices

- Minima of E are called *vortices*. e.g. Vortices where $E = 0$ are called *vacua* and have the form,

$$A_i = \partial_i \chi$$
$$\phi = \sqrt{\tau} e^{i\chi}$$

for some $\chi : \mathbb{R}^2 \to \mathbb{R}$.

- Finite energy vortex $\implies \lim_{r\to\infty} \overline{\phi}\phi = \tau$ where $r^2 = \left\|x^i\right\|^2$
- Suppose for r large, $\phi = \sqrt{\tau}\xi(\theta)$, $\xi : S^1 \to \mathrm{U}(1)$.
- Then setting $A_i = \xi(\theta)^{-1}\partial_i\xi(\theta)$ for large r yields a finite energy vortex.

Vortex Field Equations

- Can rewrite E as,

$$E = \frac{1}{2}\int \left(B - \frac{1}{2}(\tau - \overline{\phi}\phi)\right)^2 + \overline{(D_1 + \mathrm{i}D_2)\phi}(D_1 + \mathrm{i}D_2)\phi$$
$$+\tau B - \mathrm{i}(\partial_1(\overline{\phi}D_2\phi) - \partial_2(\overline{\phi}D_1\phi))\,\mathrm{d}^2x$$

- So E bounded below by $\pi\tau N$, where N is the *winding number* of $\xi(\theta)$.
- *Vortex field equations*:
  - $(D_1 + \mathrm{i}D_2)\phi = 0$ (Generalised Cauchy-Riemann equation)
  - $B - \frac{1}{2}(\tau - \overline{\phi}\phi) = 0$ (Curvature constraint)

Gauge Symmetry

- Let $M^N = \{\text{sols. of the vortex eqs. with } V = \pi\tau N\}$.
- $\mathcal{G} = C^\infty(\mathbb{R}^2, \mathsf{U}(1))$ acts (freely) on M^N as,

$$\alpha \cdot (A_i, \phi) = (A_i + \partial_i \alpha, e^{\mathrm{i}\alpha}\phi)$$

  for $e^{\mathrm{i}\alpha} \in C^\infty(\mathbb{R}^2, \mathsf{U}(1))$.
- Let $\mathcal{M}^N = M^n/\mathcal{G}$ be the *moduli space of N-vortices.*
- e.g. $\mathcal{M}^0 = \{[0, \sqrt{\tau}]\}$

Structure of the Moduli Space

- For any set of points $\{p_1, \ldots, p_N\} \subset (\mathbb{R}^2)^N / S_N$, $\exists! [A_i, \phi] \in \mathcal{M}^N$ such that $\phi^{-1}(0) = \{p_1, \ldots, p_N\}$ (Jaffe & Taubes, 1980).
- Vortices are completely determined by the position of their zeroes.
- Can identify $\mathcal{M}^N \cong (\mathbb{R}^2)^N / S_N \cong \mathbb{C}^N / S_N \cong \mathbb{C}^N$.
- In turn, can determine a Riemannian metric on $\mathcal{M}^N$ (the *vortex metric*).

Vortices on the Sphere

- Let $\Sigma = U_1 \cup U_2$ be a round sphere of radius R.
- Fix $t : U_1 \cap U_2 \to \mathrm{U}(1)$ and consider (A_i, ϕ) defined on U_1, U_2 separately such that:
  - $A_i\big|_{U_1} = A_i\big|_{U_2} - (\partial_i t)t^{-1}$ and
  - $\phi\big|_{U_1} = t\phi\big|_{U_2}$.
- $U_1 \cap U_2 \simeq S^1 \implies t$ has a well defined winding number N.
- New field equations:
  - $(D_1 + \mathrm{i}D_2)\phi = 0$
  - $\frac{1+r^2}{R} B = \tau - \overline{\phi}\phi$

The Moduli Space of Vortices on a Sphere

- Integrating the second equation yields,

$$\mathcal{M}^N \neq \emptyset \implies 4\pi N \leq \tau|\Sigma|$$

- Let $\varepsilon = \tau|\Sigma| - 4\pi N$. Then
    - $\varepsilon > 0 \implies \mathcal{M}_\varepsilon^N \cong \mathbb{CP}^N$
    - $\mathcal{M}_0^N \cong \{\text{pt}\}$
- Identify $\mathcal{M}_\varepsilon^N$ with $\{\text{Holomorphic sections on } S^2\}$, $\mathcal{M}_\varepsilon^N \cong \mathbb{CP}^N, \forall \varepsilon \geq 0$.
- Interested in how the metric on $\mathcal{M}_\varepsilon^N$ change as $\varepsilon \to 0$
- In particular, does the vortex metric tend to the “natural” metric on $\mathbb{CP}^N$ as $\varepsilon \to 0$?

Current Results and Directions

- Original conjecture made in (Baptista & Manton, 2003).
- C^0 convergence for a slight generalisation proven by Speight and Garcia (to appear).
- Currently trying to show metrics converge in C^1.

References

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